Heart of the Four Color Theorem REDUCIBILITY

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What is a graph coloring?

Convert a region coloring problem to a graph coloring problem.



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What is a graph coloring?

Problem came forth from coloring world maps.



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The Four Color Theorem

Theorem

Every planar graph can be colored in at most four colors.

A simple statement. First formulated in 1852, proven over a hundred years later in 1976.

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The Five Color Theorem

Theorem

Every planar graph can be colored in at most five colors.

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The Five Color Theorem

Theorem

Every planar graph can be colored in at most five colors.

Proof.

Every planar graph G has a vertex of $deg(v) \leq 5$.

- If $deg(v) \le 4$, can use fifth color.
- If deg(v) = 5, can always free up a color.

Remove v from the graph and repeat until there are no vertices left. Add back and color these vertices until we obtain a coloring of G.

The Five Color Theorem

Most important argument

If deg(v) = 5, can always free up a color.

- If there is an *ac*-chain, then we have isolated *b*. Therefore, there can not be a *bd*-chain from *b* to *d*. We may flip the *bd*-chain of *b* to free up the color *d*.
- If there is no *ac*-chain, then *a* and *c* are not connected. So we may flip the *ac*-chain of *a* to free up the color *c*.



Configurations

Configurations are planar graphs.

Definition

A configuration C is *contained* in a connected planar graph G if $G \setminus C$ is connected.

Definition

A configuration C is *reducible* in a graph G if its presence implies that the 4-coloring of G can be reduced to the 4-coloring of G' with less vertices.

For the five color theorem, vertices of $deg(v) \le 5$ are all reducible configurations. Therefore we can always break down a 5-coloring problem.

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Configurations



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Fundament of the Four Color Theorem

Four Color Theorem

Every planar graph G strongly-contains a configuration C that is either k-reducible, D-reducible or C-reducible in G.



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Rings

Rings play a key role in seperating a graph in two pieces with a common border.

Definition

A ring of n vertices R_n in a planar graph G is an induced cycle of G.



Figure: Valid ring (left). Invalid ring (right).

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Ring colorings

Definition

The set of all 4-colorings of a ring R in a planar graph G is given by $\Phi(R \subset G)$ or $\Phi(G)$ if R is clear from the context.

Definition

The set of all ring colorings of R_n is given by $\Phi(n) = \Phi(R_n)$.

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Rings are reducible

A ring on its own is a reducible configuration.

Theorem

The ring R_n with $n \ge 4$ is reducible in every planar graph G.

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Rings are reducible

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Proof.

Because R_n is contained, the interior of R_n is empty. Contract two non-neighboring vertices v_1 and v_3 to a new vertex u. We obtain the smaller graph G'. To reverse a coloring of G', we give v_1 and v_3 the same color as u. We obtained a coloring for G.



Ring configurations

Definition

A planar graph $C = R_n + K$ consisting of a ring R_n and an interior K is called a *ring configuration* on R_n . K is called the *core* of C.



We have already shown reducibility if $\mathcal{K} = \emptyset$ (nothing)!

Finding common ring colorings

We split our graph into M + R and $\mathcal{K} + R$. Now we try to find a common coloring on the ring R.





Such a coloring might not always be guaranteed.

Reducers

We can add extra vertices on the other side of the ring. This restricts the possible colorings we can encounter.



Figure: A graph that supports all ring colorings (left). A graph that only supports 3-colorings (right).

The ring + extra vertices and edges is a *reducer* graph.

Reducers

We restrict the size of the reducers such that $|S - R| \le k$. By increasing k, we get more control but a larger graph to color.

Reducer Size

- If k = 0, then S = R and we reduce to M + R and $\mathcal{K} + R$. This is the smallest possible reduction and ideal situation.
- If k = 1, then we can set S = R + v. Our reduced graph is then one vertex larger.
- If $k \ge |\mathcal{C}|$, then there is no point in reducing, since we obtain the same graph $M + \mathcal{C}$.

k-Reducibility

Definition

A configuration C = K + R is *k*-reducible if for all planar graphs G = M + C and some reducers S and S' on $\leq k < |C|$ vertices, there exists a common ring coloring for M + S and K + S'.

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k-Reducibility

Definition

A configuration C = K + R is *k*-reducible if for all planar graphs G = M + C and some reducers S and S' on $\leq k < |C|$ vertices, there exists a common ring coloring for M + S and K + S'.

Definition

The ring R_n is *k*-reducible if all configurations C on R_n are at most *k*-reducible.

k-reducibility

Example

The rings R_2 and R_3 are 0-reducible. The only colorings are *ab* and *abc*. Therefore, there is always a common ring coloring for M + R and $\mathcal{K} + R$.

We will be proving the 0-reducibility of R_4 and the 1-reducibility of R_5 after introducing Kempe-chains.

Kempe-chains

Definition

Let $G_{ab}(x)$ be the subgraph consisting of all the vertices colored ab in the coloring x of G. Then the Kempe-chain $\kappa_{ab}(v)$ or ab-chain of the vertex v is the component of $G_{ab}(x)$ that contains v.



Figure: We write $v_1 \stackrel{ab}{\frown} v_2$ to indicate v_1 and v_2 lie on the same *ab*-chain.

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Ring schemes

Definition

Given a coloring x of a planar graph G and the colors on its ring x(R). The scheme on R of x consists of x(R) with knowledge whether $u \in \kappa_{ab}(v)$ for two ring vertices $u, v \in R$ and colors ab.



Figure: The edge with \gg indicates order of vertices in the coloring.

Schemes imply new colorings

Definition

Given two schemes x and y. We say that x implies y if x = y or y can be obtained from x by flipping a Kempe-chain. Write $x \implies y$.

$$a \stackrel{a}{b} c a b \implies a \stackrel{a}{b} c a d$$
, $a \stackrel{d}{b} c a b \implies a \stackrel{d}{d} c a b$.

This is the key argument we used in the five color theorem!


We will be back in 5 minutes.

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Theorem

The ring R_4 is 0-reducible

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Theorem

The ring R_4 is 0-reducible

Proof. We will show that there is a common ring coloring for any $M + R_4$ and C. Let the ring colorings of the two graphs be given by

$$\mathsf{I} = \Phi(M + R_4)$$
 and $\mathsf{II} = \Phi(\mathcal{C}).$ (1)

Theorem

The ring R_4 is 0-reducible

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$$I = \Phi(M + R_4)$$
 and $II = \Phi(C)$. (1)

The situation is sketched below.



Both parts have a plain ring R_4 , which we have shown to be reducible. Therefore, we may contract any two opposing vertices.

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Both parts have a plain ring R_4 , which we have shown to be reducible. Therefore, we may contract any two opposing vertices.

This gives us a guarantee on two colorings in both sets I and II.

$$\begin{cases} abab \text{ or } abac, \\ baba \text{ or } baca \end{cases} \subset I, II.$$
 (2)

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$$\begin{cases} abab \text{ or } abac, \\ baba \text{ or } baca \end{cases} \subset I, II.$$
 (2)

Therefore, we have 4 possible options for guaranteed colorings in I and II, 3 of which are unique.

$$\widehat{1} = \{abab\}, \quad \widehat{2} = \left\{\begin{array}{c}abab\\baca\end{array}\right\}, \quad \widehat{3} = \left\{\begin{array}{c}abac\\baba\end{array}\right\}, \quad \widehat{4} = \left\{\begin{array}{c}abac\\baca\end{array}\right\}. \quad (3)$$

$$(1) = \{abab\}, \quad (2) = \begin{cases} abab\\ baca \end{cases}, \quad (3) = \begin{cases} abac\\ baba \end{cases}, \quad (4) = \begin{cases} abac\\ baca \end{cases}.$$

We consider all pairs that can occur for I and II. All but one pair already have a common ring coloring.

$$(1) = \{abab\}, \quad (2) = \begin{cases} abab\\ baca \end{cases}, \quad (3) = \begin{cases} abac\\ baba \end{cases}, \quad (4) = \begin{cases} abac\\ baca \end{cases}.$$

We consider all pairs that can occur for I and II. All but one pair already have a common ring coloring.

The only pair that does not directly have a common coloring is (1) and (4).

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Let $\{abab\} \subset I$ and $\{abac, baca\} \subset II$ be guaranteed colorings.

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Let $\{abab\} \subset I$ and $\{abac, baca\} \subset II$ be guaranteed colorings.

Suppose that in the coloring I(*abab*) we have $v_1 \stackrel{ad}{\frown} v_3$.

$$I(abab) = \stackrel{d}{a \ b \ a \ b} \implies II(abac)$$

$$I(abab) = \stackrel{d}{a \ b \ a \ b} \implies I(abcb) = II(baca).$$
(4)

Therefore, the guaranteed colorings always lead to a common coloring \Box .

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Examples of reducible configurations on R_4



Because R_4 is 0-reducible, the interior of **any** configuration on R_4 can be removed.

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Theorem

The ring R_5 is 1-reducible

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Theorem

The ring R_5 is 1-reducible

Proof. We will show that there is a common ring coloring for any M + S and $\mathcal{K} + S'$.

Let the ring colorings of the two graphs again be given by

$$I = \Phi(M + S)$$
 and $II = \Phi(\mathcal{K} + S').$ (5)

We can choose to reduce with any S and S' on ≤ 1 extra vertices. Each choice gives us guaranteed colorings for I and II.

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First we examine the guaranteed colorings without using a reducer.



Figure: The reductions $M + R_5$ and $\mathcal{K} + R_5$.

These plain rings R_5 in each graph may be further reduced by contracting two opposing vertices.

We can contract two opposing vertices of R_5 in 5 different ways. Each choice guarantees a coloring where two vertices are colored the same.

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We can contract two opposing vertices of R_5 in 5 different ways. Each choice guarantees a coloring where two vertices are colored the same.

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We can contract two opposing vertices of R_5 in 5 different ways. Each choice guarantees a coloring where two vertices are colored the same.

The *-colors are still unknown, but we are guaranteed that the colorings are possible, therefore $\Phi^{\star}\subset I,II.$

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Next, we examine the guaranteed colorings with a reducer that has 1 extra vertex.

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Next, we examine the guaranteed colorings with a reducer that has 1 extra vertex.



Figure: The reductions M + S and $\mathcal{K} + S'$.

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Next, we examine the guaranteed colorings with a reducer that has 1 extra vertex.



Figure: The reductions M + S and $\mathcal{K} + S'$.

These reducers guarantee one 3-coloring of the ring.

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These reducers guarantee **one** 3-coloring of the ring R_5 .

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These reducers guarantee **one** 3-coloring of the ring R_5 .

 $\Phi^{S} = \{\underline{c}abab \text{ or } a\underline{c}bab \text{ or } ab\underline{c}ab \text{ or } aba\underline{c}b \text{ or } abab\underline{c}\}.$ (7)

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 (7)

In total, we are guaranteed of two sets of colorings for I and II.

$$\Phi^{\star}, \Phi^{S} \subset \mathsf{I}, \mathsf{II}.$$

Next, we show that these guaranteed colorings are sufficient to find a common coloring in I and II.

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The 3-colorings are important because of two key properties that they have.

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The 3-colorings are important because of two key properties that they have.

Definition

The uniquely-colored vertex of a 3-coloring of R_5 is called the *marked* vertex, indicated by an underline such as in <u>cabab</u>.

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Definition

The uniquely-colored vertex of a 3-coloring of R_5 is called the *marked* vertex, indicated by an underline such as in <u>cabab</u>.

Definition

Two 3-colorings of R_5 are called *adjacent* if they have adjacent marked vertices, such as in <u>cabab</u> and <u>acbab</u>.

Both sets I and II are guaranteed to have one 3-coloring from Φ^{S} . There are three cases that can occur.

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- I and II have an adjacent coloring (<u>cabab</u> and <u>acbab</u>).
- I and II have a non-adjacent coloring (<u>cabab</u> and <u>abcab</u>).

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- I and II have an adjacent coloring (<u>cabab</u> and <u>acbab</u>).
- I and II have a non-adjacent coloring (<u>cabab</u> and <u>abcab</u>).
- I and II have a coloring with the same marked vertex (<u>cabab</u> and <u>dcbcb</u>). These are already equal, so we are done.

- I and II have an adjacent coloring (<u>cabab</u> and <u>acbab</u>).
- I and II have a non-adjacent coloring (*cabab* and *abcab*).

We have two lemmas to deal with these cases. Together they guarantee a common coloring in I and II.

$$\widehat{2} \implies \widehat{1}$$
 or common coloring (Lemma 1)
 \downarrow
 $\widehat{1} \implies$ common coloring (Lemma 2)

Lemma 1

If I and II have a non-adjacent coloring, then they either have an adjacent coloring or a common coloring.

The second case results in a coloring adjacent to $I(\underline{c}abab)$ as desired.

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Lemma 1

If I and II have a non-adjacent coloring, then they either have an adjacent coloring or a common coloring.

Proof. Assume we have two non-adjacent colorings $I(\underline{c}abab)$ and $II(ab\underline{c}ab)$. Suppose that $v_3 \stackrel{bc}{\frown} v_5$ in $II(ab\underline{c}ab)$. This leads to

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$$II(ab\underline{c}ab) = a \ b \ c \ a \ b \implies II(abcdb),$$

$$II(ab\underline{c}ab) = a \ b \ c \ a \ b \implies II(a\underline{c}bab).$$
(8)

The second case results in a coloring adjacent to $I(\underline{c}abab)$ as desired.
The first case results in II(*abcdb*).

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The first case results in II(abcdb). Consider the coloring I(*b**b).

The two adjacent *-colors must be different from each other and b, therefore we may assume that we have I(*bcdb). The last *-color reveals 3 possibilities.

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The first case results in II(*abcdb*). Consider the coloring I(*b**b).

The two adjacent *-colors must be different from each other and b, therefore we may assume that we have I(*bcdb). The last *-color reveals 3 possibilities.

Therefore we obtain either a common coloring or an adjacent coloring \Box .

Lemma 2

If I and II have an adjacent coloring, then they have a common coloring.

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Lemma 2

If I and II have an adjacent coloring, then they have a common coloring.

Proof. Assume we have two adjacent colorings $I(\underline{c}abab)$ and $II(\underline{ac}bab)$. Suppose that $v_3 \stackrel{bd}{\frown} v_5$ in $II(\underline{ac}bab)$. This leads to

Lemma 2

If I and II have an adjacent coloring, then they have a common coloring.

Proof. Assume we have two adjacent colorings $I(\underline{c}abab)$ and $II(\underline{a}\underline{c}bab)$. Suppose that $v_3 \stackrel{bd}{\frown} v_5$ in $II(\underline{a}\underline{c}bab)$. This leads to

$$II(a\underline{c}bab) = a c \overrightarrow{b a b} \implies I(\underline{c}abab).$$
(10)
$$II(a\underline{c}bab) = a c \overrightarrow{b a b} \implies II(acdab).$$

The first case leads to a common coloring as desired

The second case results in II(acdab). Consider the coloring I(a * * a *).

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The second case results in II(*acdab*). Consider the coloring I(a**a*). We may again assume to have I(*acda**). Then we have 3 remaining possibilities for the *-color.

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The second case results in II(*acdab*). Consider the coloring I(a**a*). We may again assume to have I(*acda**). Then we have 3 remaining particulation for the scalar.

possibilities for the *-color.

$$\begin{array}{ll} \mathsf{I}(\mathit{acdab}) &= & \mathsf{II}(\mathit{acdab}) \text{ from case 2,} \\ \mathsf{I}(\mathit{ac\underline{d}ac}) &= & \mathsf{shifted} + 2 & \mathsf{I}(\underline{c}\mathit{abab}), \\ \mathsf{I}(\mathit{a\underline{c}dad}) &= & & \mathsf{II}(\mathit{a\underline{c}bab}). \end{array}$$

Only the second case does not lead to a common coloring.

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We can repeat the same argument to continuously shift the marked vertex +2 to the right for I and II. This results in a pattern.

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At iteration 5, we obtain that II and I have the same marked vertex v_1 . Therefore, by repetition we finally obtain that

$$\mathsf{II}(\underline{ac}bab) \to \mathsf{II}(\underline{abac}b) \to \mathsf{II}(\underline{c}abab) = \mathsf{I}(\underline{c}abab) \quad \Box. \tag{11}$$

Lemma 1 and Lemma 2 together guarantee a common ring coloring. This finishes the proof \Box .

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Examples of reducible configurations on R_5 .



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Example of non-reducible configurations on R_5 .



Figure: Replacing this configuration by our reducer yields the same graph. Therefore, this configuration does not fall under the 1-reducibility of R_5 . If it were, then the four color theorem would proven.



We will be back in 5 minutes.

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Conclusion

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The Birkhoff Diamond



Figure: The Birkhoff Diamond (bir).

The degrees of vertices in the core ${\mathcal K}$ fully determines the ring around it.

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The Birkhoff Diamond



Figure: The Birkhoff Diamond (bir).

The degrees of vertices in the core \mathcal{K} fully determines the ring around it.

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$$\longrightarrow$$
 deg(v) = 5
• \longrightarrow deg(v) = 6
• \longrightarrow deg(v) = 7
 \square \longrightarrow deg(v) = 8

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The Birkhoff Diamond

| Φ(6) | | | |
|--------|--------|--------|--------|
| ababab | abacbd | abcadc | abcdab |
| ababac | abacdb | abcbab | abcdac |
| ababcb | abacdc | abcbac | abcdad |
| ababcd | abcabc | abcbad | abcdbc |
| abacab | abcabd | abcbcb | abcdbd |
| abacac | abcacb | abcbcd | abcdcb |
| abacad | abcacd | abcbdb | abcdcd |
| abacbc | abcadb | abcbdc | |
| 31 | | | |

Figure: All unique ring colorings of R_6 . The colorings of $\Phi(Bir \Diamond)$ in green.

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Consider a chain $v_4 \stackrel{bd}{\frown} v_6$ in *ababab*.

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Consider a chain $v_4 \stackrel{bd}{\frown} v_6$ in *ababab*.

Therefore, the coloring *ababab* implies the set

$$ababab \implies \{ababcb, ababac\} \subset \Phi(\mathsf{Bir}\Diamond).$$
 (13)

We say that *ababab* is a *fixable* ring coloring of Bir \Diamond .

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Definition

A coloring x implies a set of colorings II if every scheme x^* of x implies a coloring $y \in II$. Write $x \implies II$.

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Definition

A coloring x implies a set of colorings II if every scheme x^* of x implies a coloring $y \in II$. Write $x \implies II$.

Definition

A set of colorings I implies II if every $x \in I$ implies II. Write $I \implies II$.

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$\Phi_{5}(\mathsf{Bir}\Diamond) \implies \Phi_{4}(\mathsf{Bir}\Diamond) \implies \Phi_{3}(\mathsf{Bir}\Diamond) \implies \ldots \implies \Phi_{0}(\mathsf{Bir}\Diamond). \ (14)$

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$\Phi_{5}(\mathsf{Bir}\Diamond) \implies \Phi_{4}(\mathsf{Bir}\Diamond) \implies \Phi_{3}(\mathsf{Bir}\Diamond) \implies \ldots \implies \Phi_{0}(\mathsf{Bir}\Diamond). \ (14)$

Definition

A set of colorings I *n*-implies a set II if there exist sets B_i for 0 < i < n such that $I \implies B_{n-1}, B_i \implies B_{i-1}$ and $B_1 \implies II$. We write $I \stackrel{n}{\implies} II$.

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Example

$$\Phi_{5}(\mathsf{Bir}\Diamond) \stackrel{5}{\Longrightarrow} \phi_{0}(\mathsf{Bir}\Diamond) \tag{15}$$

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| Φ ₀ (Bir◊) | | Φ_1 | Φ2 | Φ3 | Φ4 | Φ ₅ |
|-----------------------|--------|----------|--------|--------|--------|----------------|
| ababac | abcadb | ababab | abacad | abacbd | abcabd | abcabc |
| ababcb | abcbab | ababcd | abcbdb | abcbdc | abcadc | |
| abacac | abcbac | abacab | | abcdac | abcdbc | |
| abacbc | abcbad | abcbcb | | abcdbd | | |
| abacdb | abcbcd | abcdad | | | | |
| abacdc | abcdab | | | | | |
| abcacb | abcdcb | | | | | |
| abcacd | abcdcd | | | | | |
| 16 | | 5 | 2 | 4 | 3 | 1 |

Figure: All *n*-implying sets of Bir \Diamond . Together a total of 31 colorings. Only differences are shown.

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Definition

The max-implying set $\overline{\Phi}(\mathcal{C})$ of a configuration \mathcal{C} is the largest *n*-implying set $\Phi_n(\mathcal{C})$.

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Example

Because $\Phi_5(Bir \Diamond) = \Phi(6)$, Bir \Diamond is D-reducible.

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What can we do if a ring coloring can not be fixed?

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The Bernhart Diamond



Figure: The Bernhart Diamond (Ber \Diamond). An example of a non D-reducible configuration. Why?

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C-reducibility



- fixable
- reducer fixable
- reducer unfixable
- symmetry fault

unfixable

| abababa abacada abcabada abchedub abchedu abcada abababa abacada abcabada abcada abcada abcada abcada abchedu abcada abcada abchedu abcada abchedu abcada abcada abchedu abcada abcada abchedu abcada | | | | | | |
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Reducers again

Reducers can be used to restrict the possible ring colorings. We need a reducer whose generated colorings are all fixable. We have already seen reducers when proving the 1-reducibility of R_5 !

A reducer consists of a contraction and extra interior edges and vertices.



Reducers again

Definition

A ring contraction $\sigma(v)$ on R is a map from R to the contracted ring $\sigma(R)$. Neighboring vertices of R may not be mapped to the same vertex by σ .



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Reducers again

Definition

A reducer (S, σ) of a configuration $\mathcal C$ consists of

- A contraction $\sigma(v)$ on R.
- A graph |S| < |C| whose boundary is the contracted ring $\sigma(R)$.



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Reducer for the Bernhart Diamond



The reducer generates the following type of colorings.

$$v_1 \ u_2 \ v_3 \ v_5 \ u_1 \ v_7 \ \mapsto \ v_1 \ u_2 \ v_3 \ u_2 \ v_5 \ u_1 \ v_7 \ u_1 \ .$$
 (16)

C-reducibility

Definition

A configuration C is C-reducible if $\Phi(S, \sigma) \subset \overline{\Phi}(C)$ for some reducer (S, σ) .

Example

The Bernhart Diamond (Ber \Diamond) is C-reducible. The red colorings are symmetry faults that can be shown to be fixable

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The two red colorings generated by the reducer are not fixable. Does this mean that $\mathsf{Ber} \Diamond$ is not C-reducible?

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The two red colorings generated by the reducer are not fixable. Does this mean that Ber \Diamond is not C-reducible?

Bernhart has shown in 1947 that these colorings *can* in fact be fixed. So there must be an error in $\overline{\Phi}(\text{Ber}\Diamond)$.

Consider the following colorings.

abcbacbc
$$(R1)$$
abcbadbc $(F1)$ abcbacac $(G1)$
(18)abcbdcbc $(R2)$ abcbacbd $(F1\star)$ abcbdbcb $(G2)$

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Consider the following colorings.

abcbacbc
$$(R1)$$
abcbadbc $(F1)$ abcbacac $(G1)$
(18)abcbdcbc $(R2)$ abcbacbd $(F1\star)$ abcbdbcb $(G2)$

Then we have the implications

$$R2 \implies \{G2,R1\}$$

$$\downarrow \qquad (19)$$

$$R1 \implies \{G1,F1\}$$

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Therefore, fixability of R1 and R2 depends fully on F1.

Timothy van der Valk





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The colorings F1 and F1* are symmetric. Therefore, F1 must be fixable!

Definition

A graph symmetry of G is a bijection f(v) on the vertices of G that preserves the neighbors of every vertex.

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A graph symmetry of G is a bijection f(v) on the vertices of G that preserves the neighbors of every vertex.

Definition

A symmetry fault of a configuration C is a coloring x that is not in $\overline{\Phi}(C)$, but whose symmetry $x \star = f(x)$ is.

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Definition

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Definition

A symmetry fault of a configuration C is a coloring x that is not in $\overline{\Phi}(C)$, but whose symmetry $x \star = f(x)$ is.

Bernhart mentioned the same problematic colorings in his proof. Is it a coincidence? We have not delved deeper into this problem.

Table of Contents

Part 1

- Graph coloring
- Reducible configurations
- k-reducibility
- Part 2
 - 0-reducibility of ring R_4
 - 1-reducibility of ring R₅
- 3 Part 3
 - D-reducibility
 - C-reducibility

4 Conclusion

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Conclusion

- *k*-reducibility extends upon ideas from the Five Color Theorem.
- D-reducibility extends upon k-reducibility for rings 6 and above.
- C-reducibility extends upon D-reducibility by avoiding bad ring colorings.
- Unavoidability guarantees that every planar graph has a reducible configuration.
- The Birkhoff Diamond and Bernhart Diamond illustrate D and C-reducibility.
- Symmetry faults in the Bernhart Diamond require more attention.

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Conclusion

The more advanced the concept of reducibility, the less reducible configurations are needed (A/B-reducibility is a case of C reducibility).



Figure: Number of reducible configurations on rings R_6 and higher in proofs of the four color theorem

Conclusion

The more advanced the concept of reducibility, the less reducible configurations are needed (A/B-reducibility is a case of C reducibility).



Figure: Number of reducible configurations on rings R_6 and higher in proofs of the four color theorem

Does "Perfect reducibility" exist? If so, it will be the heart of the four color theorem.

Timothy van der Valk

The End

Thank you for your attention. Questions?

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